Tip Vortex Effects on Oscillating Rotor Blades in Hovering Flight

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In rotor blade flutter studies, the airloads are usually estimated by the use of two-dimensional strip theory, and the effects of tip vortices and trailing vortices are generally neglected. The present paper examines the validity of this method of approach and gives results which reveal to what extent the assumptions made are acceptable. Compressibility and frequency effects are taken fully into account, but curvature effects resulting from the helical nature of the wake are ignored. The latter are assumed to be negligible when the frequency of oscillation of the blades is such that each blade of the rotor oscillates at several cycles per rotation. From the general theory developed, it is concluded that up to tip Mach numbers of about 0.8, the use of two-dimensional strip theory would not lead to serious error provided the blades oscillate at several cycles per rotation. This general conclusion is derived on the basis of a comparison of the downwash induced at points along the blade as given by the usual two-dimensional theory and the corresponding values derived by an improved method which takes tip effects into account. In the more accurate theory, the two-dimensional mathematical model of the flow normally adopted is replaced by a modified version in which the trailing vortices are fully represented.

Nomenclature

l = half-chord
U,M = local velocity and Mach number
x = lX coordinate transformation
y = lY/β, coordinate transformation
z = lZ/β, coordinate transformation
t = lT/U,
$\omega(= pl/U)$ = frequency parameter
ν = ω/β^2
κ = $M\nu$
λ = $M^2\nu$
β = $(1 - M^2)^{1/2}$
$w(= w'e^{ipt})$ = downwash distribution
$\phi(= l\Phi e^{i(\lambda X+\omega T)})$ = velocity potential
$lk(= \phi_a - \phi_b)$ = discontinuity in velocity potential across vortex sheet
$K(= \Phi_a - \Phi_b)$ = discontinuity in modified potential
Q = number of blades of rotor
q = blade number ($q = 0,1,2,3, \ldots Q - 1$)
Ω = angular velocity of blade rotation
$\epsilon(= p/\Omega)$ = frequency ratio
$\psi_0(= 2\pi\epsilon)$ = phase lag per rotation
ψ_q = phase lead of qth blade
$dl(\equiv lD/\beta)$ = spacing of vortex sheets
$X(= -\cos\theta)$ = for points on the reference blade

$\Gamma_n, \bar{K}_n$ *Distributions*

$\Gamma_n = i\nu\bar{K}_n + \partial\bar{K}_n/\partial X$
$\bar{K}_n = e^{-i\nu X}\int_{-1}^{X}\Gamma e^{i\nu X}dX$
$\Gamma_0 = 2[C(\nu)\cot(\theta/2) + i\nu\sin\theta]$
$\Gamma_1 = -2\sin\theta + \cot(\theta/2) + i\nu(\sin\theta + \sin2\theta/2)$
$\Gamma_n = -2\sin2\theta + i\nu[\sin(n+1)\theta/(n+1) - \sin(n-1)\theta/(n-1)] \ldots n \geq 2$
$\bar{K}_0 = 2\pi X_0(\nu)e^{-i\nu X} \ldots X \geq 1$
$\bar{K}_1 = \sin\theta + \sin2\theta/2$
$\bar{K}_n = \sin(n+1)\theta/(n+1) - \sin(n-1)\theta/(n-1) \ldots n \geq 2$

$C(\nu)$ = $H_1^{(2)}(\nu)/[H_1^{(2)}(\nu) + iH_0^{(2)}(\nu)]$
$X_0(\nu)$ = $C(\nu)J_0(\nu) + i[1 - C(\nu)]J_1(\nu)$
J_0,J_1 = Bessel functions
K_0,K_1 = modified Bessel functions

Received December 9, 1969; revision received May 27, 1970. The authors are greatly indebted to Mrs. Frances Todd for her assistance in the preparation of this paper for publication. Work supported by the U.S. Army Research Office-Durham under Project Themis Contract DAHCO4-69-C-0015.

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1. Introduction

In an earlier paper[1] a study was made of compressibility effects on an oscillating rotor blade in hovering flight. A two-dimensional mathematical model of the flow was used and the oscillating blade and its helical wake were represented simply by an airfoil and its immediate wake together with an array of infinite regularly spaced vortex sheets as shown in Fig. 1. Though this model is very idealized in that it ignores the helical nature of the wake, it has nevertheless been widely used in helicopter flutter studies and has been of value in establishing the causes of particular types of instability.[2,3] However, for detailed investigation of the flow near the tip of the blade, a two-dimensional representation is likely to be inadequate since tip vortex effects are ignored. In the present paper a method is outlined for taking such effects into account.

Firstly, it is assumed that the inflow field produced by rotation of the blades in steady motion is known and that the geometry of the helical wake of each blade of the rotor is defined. Any small oscillation of the rotor blades would perturb this flow, but in the development of the theory it is assumed that the free vorticity arising from such oscillations is distributed over the known steady helical wake. The airloads due to these perturbations would be additional to those for steady blade motion. For lightly loaded blades the steady inflow distribution is usually regarded as being

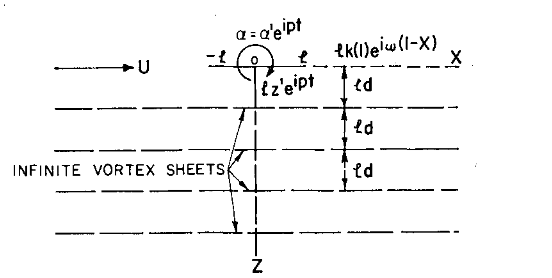

Fig. 1 Mathematical model of airfoil and vortex sheets.

fairly uniform, but for heavily loaded blades the inflow can vary appreciably with radial distance as shown in Refs. 4 and 5. In the present analysis, it is assumed initially that the blades are designed so as to give approximately uniform inflow distribution and that the trailing tip vortex is not strong enough to distort greatly the regular form of the helical wake, or to cause severe contraction. Under these conditions, an improved model of the flow can be developed for determining the perturbation effects due to blade oscillations. Let it be assumed that the blade radius is large so that, for points near the tip, the induced downwash resulting from the trailing vortices can be calculated as if they were straight and the effect of curvature negligible. The full two-dimensional model of the flow can then be replaced by a 'half-model' extending from the tip inwards towards a distant hub as shown in Fig. 2. To calculate the downwash at a particular section of the oscillating blade, the chordwise vorticity distribution over any section of the reference blade is taken to be the same as that for the section under consideration. It is also supposed that the blade is moving effectively without rotation at a speed $U(= r\Omega)$, where U is the local velocity at the section considered. For the purpose of calculating the downwash distribution over the section $y = y_1$, the helical wake is represented by a system of transverse and trailing vortices lying in parallel planes, the regular spacing between the planes being denoted by $ld(y_1)$ as indicated in Fig. 2. In general, the plane $x = 0$ will cut the helical wake in a set of almost parallel curves. For a six-bladed rotor, when the blades are heavily loaded, these curves are distorted near the tip and the spacing between the reference blade and the first curve is smaller than the distance between the individual curves as shown in Refs. 4 and 5. However, in the present study, the spacing between consecutive wakes is assumed to be constant and equal to the local value at the section $y = y_1$ at which the perturbed downwash distribution is to be calculated. It is shown later that the analysis could easily be modified to allow for unequal spacing between the wakes if this were considered necessary. When the downwash field induced by the system of wakes is known, the additional airloads resulting from blade vibration can be calculated and the effect of frequency, Mach number, and blade phasing can be determined. The results of such calculations will be given in a further paper.

The present study is concerned only with disturbances in hovering flight but it should be mentioned that unsteady airloads for forward flight have been calculated by several authors, notably by Miller,[6-8] in a series of important papers, and by Piziali.[9]

2. Development of General Theory

The analysis given in this paper follows closely that of Ref. 1, except that the theory is developed on a three-dimensional rather than a two-dimensional basis to allow for the effect of trailing vortices. With the coordinate system illustrated in Fig. 2 and after the introduction of new coordinates, X,Y,Z,T so that

$$x = lX,\ y = lY/\beta,\ z = lZ/\beta,\ t = lT/U \tag{1}$$

it can be shown that the velocity potential ϕ of the disturbed flow resulting from the oscillating blade can be defined in terms of a modified potential Φ given by

$$\phi = l\Phi e^{i(\lambda X+\omega T)} \tag{2}$$

where $\lambda = M^2\omega/(1 - M^2)$, $\omega = pl/U$, and p is the frequency of the oscillation in radians per second. This modified potential is such that it satisfies the wave equation

$$\nabla^2\Phi + \kappa^2\Phi = 0 \tag{3}$$

where $\kappa \equiv M\omega/(1 - M^2)$.

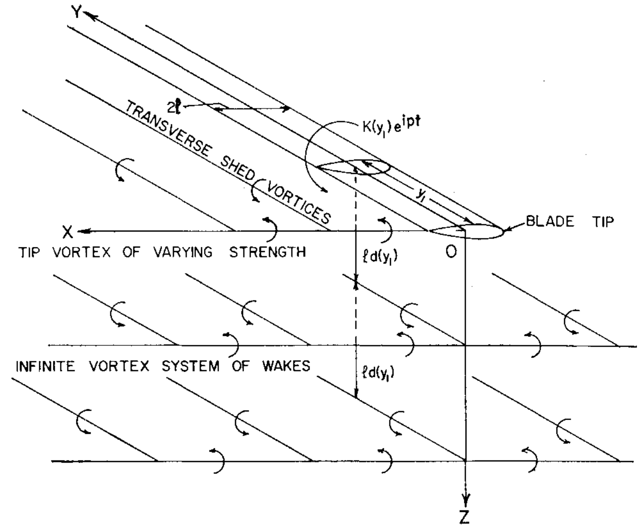

Fig. 2 Half model representation with tip vortex.

Let ϕ_a, ϕ_b be the values of ϕ above and below a vortex (or doublet) sheet, respectively, and let us assume that corresponding values of the modified potential Φ are Φ_a and Φ_b. Then, if

$$lk(x,y) \equiv \phi_a - \phi_b \text{ and } K(X,Y) \equiv \Phi_a - \Phi_b \tag{4}$$

we have the following relation between the discontinuity in the velocity potential and that in the modified potential, namely

$$k(x,y) = K(X,Y)e^{i(\lambda X+\omega T)} \tag{5}$$

The $k(x,y)$ and $K(X,Y)$ distributions are in effect doublet distributions over the blade and its wake. The corresponding lift distribution is given by Euler's equations of motion as

$$\bar{l}(X,Y) = \rho U(i\omega k + \partial k/\partial X) = \rho U\Gamma e^{i(\lambda X+\omega T)}$$
$$\text{with} \quad \Gamma \equiv i\nu K + \partial K/\partial X \text{ and } \nu \equiv \omega/(1 - M^2) \tag{6}$$

Since the wake can sustain no lift, we must have

$$\Gamma = i\nu K + \partial K/\partial X = 0 \tag{7}$$

when $X > 1$. It then follows that in the immediate wake of the blade

$$K(X,Y) = K(1,Y)e^{i\nu(1-X)} \tag{8}$$

where $K(1,Y)$ denotes the value of $K(X,Y)$ at the trailing edge, $X = 1$, of section Y. In the subsequent analysis this is redefined for convenience as

$$K(X,Y) = K_{00}(X_1,Y)e^{-i\nu\xi} \tag{9}$$

where $\xi \equiv X - X_1$ and $X = X_1$ is an arbitrary reference line along which the downwash distribution induced by the system of wakes is to be determined.

The K distributions for the individual wakes of the infinite system below the blade can be defined similarly. Let the rotor have Q blades, denoted by $q = 0,1,2,3, \ldots Q - 1$, respectively, where $q = 0$ refers to the reference blade. If the motion of the q'th blade is identical with that of the reference blade except for a phase lead ψ_q, the corresponding K distribution of the wake of this blade after n revolutions will be defined by

$$K_{qn}(X,Y) = K_{00}(X_1,Y)e^{-i\nu\xi-2\pi i\epsilon(n+q/Q)+i\psi_q} \tag{10}$$

where $\epsilon \equiv p/\Omega$, p being the frequency of oscillation and Ω the angular rate of rotation.

The K distribution over the wake arising from the reference blade after n revolutions is simply denoted by

$$K_{on}(X,Y) = K_{00}(X_1,Y)e^{-i\nu\xi-2\pi i\epsilon n} \tag{11}$$

since $q = 0$ and $\psi_q = 0$ for this case. The immediate wake of the reference blade has already been given in Eq. (9) and corresponds to $n = 0$ in Eq. (11). It is evident from these formulas that all the wake distributions will be specified when $K_{00}(X_1,Y)$ is determined. From Eqs. (9) and (11), it follows that

$$K_{00}(X_1,Y) = K(1,Y)e^{i\nu(1-X_1)} \tag{12}$$

and this can only be defined when $K(X,Y)$ and hence $K(1,Y)$ over the reference blade is known. To obtain the appropriate distribution, use can be made of the techniques employed in unsteady compressible flow theory[10,11] as illustrated below.

Let it be assumed that the reference blade is oscillating in such a way as to have a prescribed normal velocity distribution $w(X,Y,T)$ $(\equiv w'(X,Y)e^{i\omega T})$. Then it follows from Eq. (2) that

$$\partial\phi/\partial z = \beta(\partial\Phi/\partial Z)e^{i(\lambda X+\omega T)} = w(X,Y,T) \tag{13}$$

and that

$$W(X,Y) = \partial\Phi/\partial Z = w'(X,Y)e^{-i\lambda X}/\beta \tag{14}$$

where $W(X,Y)$ is the amplitude of the corresponding modified downwash over the blade. Since the vorticity distributions over the blades and the wakes must be such that the corresponding induced downwash is equal to the prescribed normal velocity over the reference blade, the following condition for tangential flow over the blades surface must be satisfied, namely,

$$W(X,Y) = W_b(X,Y) + W_i(X,Y) \tag{15}$$

In Eq. (15), $W(X,Y)$ is assumed to be known. $W_b(X,Y)$ is the modified downwash amplitude resulting from the vorticity distribution over the reference blade and its immediate wake and $W_i(X,Y)$ is the corresponding distribution induced by the system of wakes underneath the blade considered. From unsteady compressible flow theory,[10,11] it is known that

$$4\pi W_b(X_1,Y_1) = \int_0^\infty \int_{-1}^\infty K(X,Y)\frac{\partial^2}{\partial Z_1{}^2}(e^{-i\kappa r}/r)dXdY \tag{16}$$

where $r \equiv [(X - X_1)^2 + (Y - Y_1)^2 + Z_1{}^2]^{1/2}$ and $Z_1 \to 0$. In the wake, the form of $K(X,Y)$ is given by Eq. (8) and it satisfies the wake condition represented by Eq. (7). In the absence of the system of wakes $W_i = 0$, and, by Eq. (15), W_b must be equal to the prescribed modified normal velocity amplitude W. When W_b is known, the corresponding K distribution can be found by solving Eq. (16) by methods already developed.[11]

For the problem under consideration, however, the modified downwash W_i induced by the system of wakes is not zero and it must be taken into account. Instead of Eq. (16), we then have to solve the equation

$$4\pi[W(X_1,Y_1) - W_i(X_1,Y_1)] = \int_0^\infty\int_{-1}^\infty K(X,Y)\frac{\partial^2}{\partial Z_1{}^2}\left(\frac{e^{-i\kappa r}}{r}\right)dXdY \tag{17}$$

where $Z_1 \to 0$. By the use of Eqs. (10) and (11), it follows that

$$4\pi W_i(X_1,Y_1) = \sum_{n=1}^{\infty} e^{-2\pi i n\epsilon}\int_0^\infty\int_{-\infty}^\infty K_{00}(X_1,Y)e^{-i\nu\xi}\frac{\partial^2}{\partial Z_1{}^2}\times\left(\frac{e^{-i\kappa r_1}}{r_1}\right)d\xi dY + \sum_{n=0}^{\infty}\sum_{q=1}^{Q-1}\exp\left[i\psi_q - \frac{2\pi i\epsilon}{Q}(nQ+q)\right]\times \int_0^\infty\int_{-\infty}^\infty K_{00}(X_1,Y)e^{-i\nu\xi}\frac{\partial^2}{\partial Z_1{}^2}\left(\frac{e^{-i\kappa r_2}}{r_2}\right)d\xi dY \tag{18}$$

where

$$r_1{}^2 \equiv \xi^2 + (Y - Y_1)^2 + (nQD - Z_1)^2 \equiv \xi^2 + \alpha_1{}^2$$
$$r_2{}^2 \equiv \xi^2 + (Y - Y_1)^2 + [(nQ + q)D - Z_1]^2 = \xi^2 + \alpha_2{}^2$$

and D has its local value at $Y = Y_1$. If necessary, wake contraction effects could be allowed for at this stage by replacing the zero lower limit of the integral with respect to Y by Y^*, where Y^* represents the inward displacement of the wake, which varies with each vortex sheet. Allowance could also be made for irregular spacing of the wakes.

3. Approximate Evaluation of $W_i(X,Y)$

With the full two-dimensional model of the flow, it is assumed that the distribution $K(X,Y)$ is independent of Y and that it has the value $K(X,Y_1)$, where $Y = Y_1$ denotes the section over which the downwash distribution is to be determined. Similarly, with the "half-model" representation adopted in this study, the $K(X,Y)$ distribution is replaced by a constant spanwise distribution $K(X,Y_1)$ extending from the tip, $y = 0$, to $y = \infty$, but which varies with X as specified by Eq. (8). Furthermore, the integrals with respect to ξ in Eq. (18) can be expressed in terms of the modified Bessel Functions K_0,K_1 as follows. When $\nu > \kappa$, which is the case when $M < 1$

$$\int_{-\infty}^\infty \frac{e^{-i\nu\xi-i\kappa r_1}}{r_1}d\xi = 2\int_0^\infty \frac{\cos\nu\xi_e{}^{-i\kappa(\xi^2+\alpha_1{}^2)^{1/2}}}{(\xi^2+\alpha_1{}^2)^{1/2}}d\xi = 2K_0(\nu\beta\alpha_1) \tag{19}$$

It can also be readily proved that

$$\partial^2K_0(\nu\beta\alpha_1)/\partial Z_1{}^2 = \nu^2\beta^2K_0(\nu\beta\alpha_1) - (\partial^2/\partial y^2)K_0(\nu\beta\alpha_1) \tag{20}$$

where, as in Eq. (18),

$$\alpha_1{}^2 \equiv \overline{Y - Y_1}^2 + (nQD - Z_1)^2$$

Hence it follows that in the limit, when $Z_1 \to 0$

$$\int_0^\infty \frac{\partial^2}{\partial Z_1{}^2}K_0\{\nu\beta[(Y - Y_1)^2 + (nQD - Z_1)^2]^{1/2}\}dY = \nu^2\beta^2\int_0^\infty K_0\{\nu\beta[(Y - Y_1)^2 + n^2Q^2D^2]^{1/2}\}dY + \frac{\nu\beta Y_1}{(Y_1{}^2 + n^2Q^2D^2)^{1/2}}K_1[\nu\beta(Y_1{}^2 + n^2Q^2D^2)^{1/2}] \tag{21}$$

The integral involving r_2 can be expressed in a similar form so that, finally, we obtain

$$2\pi W_i(X_1,Y_1) = K_{00}(X_1,Y_1)[S_1 + S_2] \tag{22}$$

where

$$S_1 = \sum_{n=1}^{\infty} e^{-2\pi i n\epsilon}\left(\frac{\nu\beta Y_1}{(Y_1{}^2 + n^2Q^2D^2)^{1/2}}K_1\times [\nu\beta(Y_1{}^2 + n^2Q^2D^2)^{1/2}] + \nu^2\beta^2\int_0^\infty K_0\{\nu\beta[(Y - Y_1)^2 + n^2Q^2D^2]^{1/2}\}dY\right)$$

and

$$S_2 = \sum_{n=0}^{\infty}\exp\left[i\psi_q - \frac{2\pi i\epsilon}{Q}(nQ+q)\right]\times \left(\frac{\nu\beta Y_1}{[Y_1{}^2 + (nQ+q)^2D^2]^{1/2}}K_1\{\nu\beta[Y_1{}^2 + (nQ+q)^2D^2]^{1/2}\} + \nu^2\beta^2\int_0^\infty K_0\{\nu\beta[(Y - Y_1)^2 + (nQ+q)^2D^2]^{1/2}\}dY\right)$$

In the subsequent analysis, let

$$\sigma = \nu\beta Y = \omega y/l,\ \mu = D\nu\beta = \omega d$$
$$f = (\sigma^2 + n^2Q^2\mu^2)^{1/2},\quad g = [\sigma^2 + (nQ+q)^2\mu^2]^{1/2}$$

and let f_1 and g_1 denote the values of f and g respectively when $\sigma = \sigma_1$. After some reduction we find that

$$S_1 = \nu\beta \sum_{n=1}^{\infty} e^{-2\pi i n\epsilon} \left[\pi e^{-nQ\mu} + \frac{\sigma_1 K_1(f_1)}{f_1} - \int_{\sigma_1}^{\infty} K_0(f)d\sigma\right] \tag{23}$$

Similarly, it can be proved that

$$S_2 = \nu\beta \sum_{n=0}^{\infty}\sum_{q=1}^{Q-1} \exp\left[i\psi_q - \frac{2\pi i}{Q}(nQ+q)\epsilon\right] \times \left[\pi e^{-(nQ+q)\mu} + \frac{\sigma_1 K_1(g_1)}{g_1} - \int_{\sigma_1}^{\infty} K_0(g)d\sigma\right] \tag{24}$$

We may then sum Eqs. (23) and (24) to obtain

$$S_1 + S_2 = (\pi/QD)[F_0(Q) + F_1(Q) + F_2(Q)] \tag{25}$$

where

$$F_0(Q) = \frac{Q\mu}{e^{\gamma}-1}\left[1 + \sum_{q=1}^{Q-1}\exp\left(i\psi_q + \frac{Q-q}{Q}\gamma\right)\right]$$

$$F_1(Q) = \frac{Q\mu}{\pi}\sum_{n=1}^{\infty} e^{-2\pi i n\epsilon}\left[\frac{\sigma_1 K_1(f_1)}{f_1} - \int_{\sigma_1}^{\infty} K_0(f)d\sigma\right]$$

$$F_2(Q) = \frac{Q\mu}{\pi}\sum_{n=0}^{\infty}\sum_{q=1}^{Q-1}\exp\left[i\psi_q - 2\pi i\epsilon\left(n + \frac{q}{Q}\right)\right] \times \left[\frac{\sigma_1}{g_1}K_1(g_1) - \int_{\sigma_1}^{\infty} K_0(g)d\sigma\right]$$

In the above formulas, $\gamma = 2\pi i\epsilon + Q\mu$ and $|e^{-\gamma}| < 1$.

From Eqs. (22) and (25) it follows that the modified downwash amplitude at the point X_1, Y_1 on the reference blade is

$$2\pi W_i(X_1,Y_1) = (\pi/QD)K_{00}(X_1,Y_1)[F_0(Q) + F_1(Q) + F_2(Q)] \tag{26}$$

By the use of Eqs. (14) and (26), the formula for the actual downwash induced by the system of wakes can be derived, namely

$$2\pi w_i(x_1,y_1) = [\pi k(l,y_1)/Qd]e^{i\omega(1-X_1)} \times [F_0(Q) + F_1(Q) + F_2(Q)] \tag{27}$$

Since the expression on the right is independent of Mach number, it follows that the downwash induced by the system of wakes for a given circulation $k(l,y)$ is the same for compressible and incompressible flow for the same value of ω. This result is due to the fact that the vortex sheets extend from $-\infty$ to ∞ and was previously noted in Ref. 1, where the full two-dimensional model of the flow excluding tip vortices was used. For that case, however, the following formula was deduced for the corresponding downwash $\bar{w}_i$, namely

$$2\pi\bar{w}_i(x_1,y_1) = [\pi k(l,y_1)/Qd]F_0(Q)e^{i\omega(1-X_1)} \tag{28}$$

The ratio of the downwash given by the half model with tip vortices included to that given by the full two-dimensional model is then

$$w_i/\bar{w}_i = 1 + (F_1 + F_2)/F_0 \tag{29}$$

As the distance from the tip is increased, w_i varies according to formula Eq. (29) and should tend to the two-dimensional value $\bar{w}_i$ when $\omega \neq 0$ and y_1 is large. A study of the variation of w_i with the parameters involved would give information on the accuracy of the two-dimensional model of the flow normally used when two-dimensional strip theory is applied to calculate the airloads on oscillating blades.

Though the preceding analysis has been developed for a multibladed rotor and the formula for $w_i/\bar{w}_i$ is quite general, only the case of a single rotor blade is examined in detail in this paper. Equation (29) then reduces to

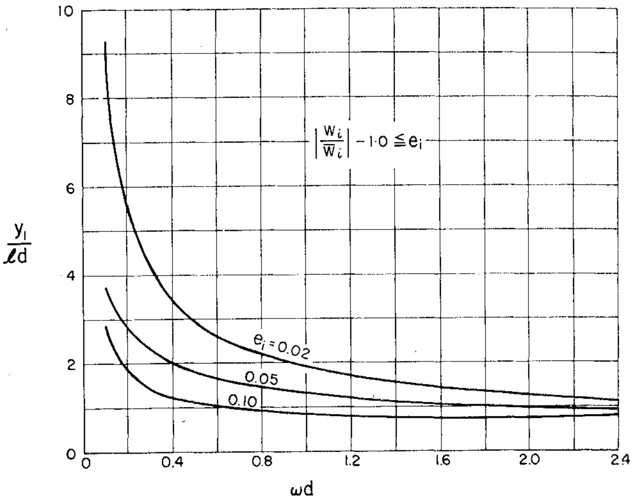

Fig. 3 Values of error, e_i.

$$w_i/\bar{w}_i = 1 + F_1(1)/F_0(1) \tag{30}$$

where

$$F_0(1) = \omega d/(e^{\gamma} - 1)$$

$$F_1(1) = \frac{\omega d}{\pi}\sum_{n=1}^{\infty} e^{-2\pi i n\epsilon}\left\{\frac{\sigma_1}{(\sigma_1^2 + n^2\omega^2 d^2)^{1/2}} K_1[(\sigma_1^2 + n^2\omega^2 d^2)^{1/2}] - \int_{\sigma_1}^{\infty} K_0[(\sigma^2 + n^2\omega^2 d^2)^{1/2}]d\sigma\right\}$$

The formula for $F_0(1)$ is only valid provided $|e^{-\gamma}| < 1$ and it should be noted that this condition is violated when the frequency of oscillation is actually zero. Curves showing the variation of $F_0(1)$ with ωd for particular values of ϵ ($\equiv p/\Omega$) are given in Ref. 1, the fractional part of ϵ only being significant in e^{γ}. Similarly, the function $F_1(1)$ can be evaluated and plotted for different values of the relevant parameters. From these results, $w_i/\bar{w}_i$ can then be determined. Typical values are shown plotted in Fig. 3 and from the curves an estimate of the inaccuracy arising from the neglect of the tip-vortices can be derived for different values of y_1/l and ω.

4. Determination of $W_b(X_1, Y_1)$

Incompressible Flow

Let us first consider the incompressible flow case when M, λ and κ are all equal to zero and $\nu = \omega$. (See Refs. 12, 13.) Equation (16) then reduces to

$$4\pi W_b(X_1,Y_1) = \int_0^{\infty}\int_{-1}^{\infty} K(X,Y)\frac{\partial^2}{\partial Z_1^2}\left(\frac{1}{r}\right)dXdY \tag{31}$$

where $Z_1 \to 0$ and $\Gamma = i\omega K + \partial K/\partial X = 0$ in the wake. In the above equation, W_b and K are the complex amplitudes of the actual downwash and circulation, respectively. The corresponding lift distribution over the chord of any section Y is given by

$$\bar{l}(X,Y) = \rho U\Gamma(X,Y) \tag{32}$$

and the spanwise lift and moment distributions can then be derived. If it is assumed that Γ and the corresponding K distribution can be represented in the form

$$\Gamma = U[C_0(Y)\Gamma_0 + C_1(Y)\Gamma_1 + C_2(Y)\Gamma_2 + \text{etc.}]$$

and

$$K = U[C_0(Y)\bar{K}_0 + C_1(Y)\bar{K}_1 + C_2(Y)\bar{K}_2 + \text{etc.}] \tag{33}$$

it can be proved that the downwash $\bar{W}_b(X,Y_1)$, as given by

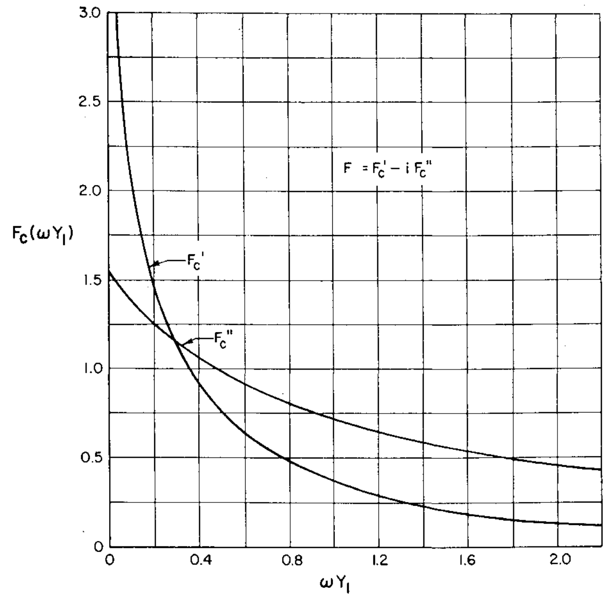

Fig. 4 Values of $F_c(\omega Y_1)$.

two-dimensional theory, is

$$\bar{W}_b(X_1,Y_1) = U[C_0(Y_1) + C_1(Y_1) \times (\tfrac{1}{2} + \cos\theta_1) + C_2(Y_1)\cos 2\theta_1 + \ldots] \quad (34)$$

where $X_1 = -\cos\theta_1$. The symbols Γ_0, Γ_1 and $\bar{K}_0,\bar{K}_1$ etc. have been defined in the list of symbols, and it should be noted that in the wake all the $\bar{K}$ distributions except $\bar{K}_0$ are zero.

If we next determine $W_b(X_1,Y_1)$ on the basis of the "half-model" representation used in this paper, we can write Eq. (31) in the form

$$4\pi W_b(X_1,Y_1) = \int_{-\infty}^{\infty}\int_{-1}^{\infty} K(X,Y_1)\frac{\partial^2}{\partial Z_1^2}\left(\frac{1}{r}\right)dXdY - \int_{-\infty}^{0}\int_{-1}^{\infty} K(X,Y_1)\frac{\partial^2}{\partial Z_1^2}\left(\frac{1}{r}\right)dXdY \quad (35)$$

where $K(X,Y)$ is assumed to have the local value $K(X,Y_1)$ all along the blade. The first term in Eqs. (35) yields the value of W_b given by two-dimensional theory while the second term represents the correction when the tip vortex is taken into account. Equations (35), after some reduction, may then be written as

$$4\pi W_b(X_1,Y_1) = 4\pi\bar{W}_b(X_1,Y_1) + \int_{-1}^{\infty}\frac{\partial K}{\partial X}\left\{\frac{1}{X - X_1} + \frac{1}{Y_1} - \frac{[(X - X_1)^2 + Y_1^2]^{1/2}}{(X - X_1)Y_1}\right\}dX \quad (36)$$

where in the wake

$$K(X,Y_1) = 2\pi U C_0(Y_1)X_0(\omega)e^{-i\omega X}, \ldots X \geq 1 \quad (37)$$

and the function $X_0(\omega)$ is defined in the list of symbols.

If we assume further that the transverse parts of the angular vortices are concentrated along the mid-chord line and consider the downwash distribution along this line, we find that approximately

$$4\pi W_b(0,Y_1) = 4\pi\bar{W}_b(0,Y_1) + \frac{K(1,Y_1)}{Y_1} - i\omega K(1,Y_1) \times \int_0^{\infty} e^{-i\omega X}\left[\frac{1}{X} + \frac{1}{Y_1} - \frac{(X^2 + Y_1^2)^{1/2}}{XY_1}\right]dX \quad (38)$$

where $\bar{W}_b$ is given by Eq. (34). The second and third terms arise from the vorticity distribution Γ_0 and the corresponding $\bar{K}_0$ and it follows that only the first term of Eq. (34) would be affected by the trailing tip vortex and shed vorticity. If we write $\bar{W}_0(0,Y_1)$ for $UC_0(Y_1)$, we see that the corresponding value given by the "half-model" representation would be approximately $W_0(0,Y_1)$ where

$$W_0(0,Y_1) = \bar{W}_0(0,Y_1) + [K(1,Y_1)/4\pi Y_1] \times [1 - i\omega Y_1 F_c(\omega Y_1)] \quad (39)$$

and

$$F_c(\omega Y_1) = \int_0^{\infty} e^{-i\omega X}\left[\frac{1}{X} + \frac{1}{Y_1} - \frac{(X^2 + Y_1^2)^{1/2}}{XY_1}\right]dX \quad (40)$$

The above integral was originally derived by Cicala[14] and its real and imaginary parts are shown plotted in Fig. 4. From Eq. (37), it also follows that

$$\bar{W}_0(0,Y_1) = K(1,Y)P(\omega)$$

where (41)

$$P(\omega) = e^{i\omega}/2\pi X_0(\omega) = (\omega/4)[H_0^{(2)}(\omega) - iH_1^{(2)}(\omega)]e^{i\omega}$$

Finally, we then have

$$W_0/\bar{W}_0 = 1 + [1 - i\omega Y_1 F_c(\omega Y_1)]/4\pi Y_1 P(\omega) \quad (42)$$

where the second term on the right is the correction resulting from tip effects.

Compressible Flow

When compressibility is taken into account, Eq. (16) must be used. If we again assume that $K(X,Y) = K(X,Y_1)$, then on the basis of the 'half-model' representation adopted in this paper, we can replace Eq. (16) by

$$4\pi W_b(X_1,Y_1) = \int_{-\infty}^{\infty}\int_{-1}^{\infty} K(X,Y_1)\frac{\partial^2}{\partial Z_1^2}\left(\frac{e^{-i\kappa r}}{r}\right)dXdY - \int_{-\infty}^{0}\int_{-1}^{\infty} K(X,Y_1)\frac{\partial^2}{\partial Z_1^2}\left(\frac{e^{-i\kappa r}}{r}\right)dXdY = 4\pi[\bar{W}_b(X_1,Y_1) - W_L(X_1,Y_1)] \quad (43)$$

In the above equation, $\bar{W}_b(X_1,Y_1)$ is the value of the downwash given by two-dimensional theory while $W_L(X_1,Y_1)$ is the modified downwash resulting from the distribution $K(X,Y_1)$ over the area of integration indicated. Since the point X_1,Y_1 is outside this area, we can assume $Z_1 = 0$ and replace r by R where

$$R = [(X - X_1)^2 + (Y - Y_1)^2]^{1/2} \quad (44)$$

It then follows from Eq. (43) that

$$4\pi W_L(X_1,Y_1) = \int_{-\infty}^{0}\int_{-1}^{\infty}\frac{K(X,Y_1)}{R}\frac{\partial}{\partial R}\left(\frac{e^{-i\kappa R}}{R}\right)dXdY \quad (45)$$

If in the definition of R, X is replaced by ξ, we can express Eq. (45) alternatively as

$$4\pi W_L(X_1,Y_1) = \int_{-\infty}^{0}\int_{-1}^{\infty}\frac{\partial K(X,Y_1)}{\partial X}\int_X^{\infty}\frac{1}{R}\frac{\partial}{\partial R}\left(\frac{e^{-i\kappa R}}{R}\right) \times d\xi\, dXdY \quad (46)$$

$$= \int_{-1}^{\infty}\frac{\partial K(X,Y_1)}{\partial X}E(X - X_1,Y_1)dX$$

where

$$E(X - X_1,Y_1) = \int_{-\infty}^{0}\int_X^{\infty}\frac{1}{R}\frac{\partial}{\partial R}\left(\frac{e^{-i\kappa R}}{R}\right)d\xi dY \quad (47)$$

is the contribution to W_L of a doublet sheet of constant unit strength extending over the area of integration indicated. In Eq. (46), the general distribution $K(X,Y_1)$ used in Eq. (45) is replaced by a number of superimposed layers of constant strength $\partial K/\partial X$ extending from X to ∞.

For incompressible, flow, $\kappa = 0$, and it can readily be deduced that

$$E(X - X_1, Y_1) = \frac{[(X - X_1)^2 + Y_1^2]^{1/2}}{(X - X_1)Y_1} - \frac{1}{X - X_1} - \frac{1}{Y_1} \tag{48}$$

With the use of Eqs. (46) and (48) it then follows that Eq. (43) would reduce to Eq. (36) from which the downwash for incompressible flow conditions can be determined.

For compressible flow, however, the double integral expression for E given in Eq. (47) is more difficult to evaluate. Let us first consider the particular integral when both X and X_1 are zero. For this case, we may then substitute $\xi = R \sin\chi$, $Y_1 - Y = R \cos\chi$ in Eq. (47) to obtain

$$E(0, Y_1) = \frac{1}{Y_1}\left\{\int_1^\infty \frac{e^{-i\kappa Y_1 q}}{q}\left[\frac{(q^2 - 1)^{1/2} - q}{q}\right] dq + \frac{\pi i}{2} \times H_0^{(2)}(\kappa Y_1) + \int_1^\infty \frac{e^{-i\kappa Y_1 q}}{q} dq\right\} \tag{49}$$

where $q = \sec\chi$. The first integral is rapidly convergent and can be evaluated numerically without difficulty. Values of the second and third terms in Eq. (49) are tabulated in Ref. 15.

The general integral is more difficult to evaluate, but if points along mid-chord only for which $X_1 = 0$ are considered, we find that

$$E(X, Y_1) = -\int_{q_s}^\infty \frac{e^{-i\kappa X q} dq}{X q^2 (q^2 - 1)^{1/2}} - \int_{q_c}^\infty \frac{e^{-i\kappa Y_1 q} dq}{Y_1 q^2 (q^2 - 1)^{1/2}} \tag{50}$$

where $q_s = \operatorname{cosec}\chi_c$, $q_c = \sec\chi_c$, and $\tan\chi_c = X/Y_1$. It should be noted that E varies with the parameter κ which involves the frequency of the oscillation and Mach number.

By the use of Eq. (50), values of $E(X, Y_1)$ can be calculated for different values of κ ($\equiv M\nu$) for a range of values of X and Y_1. Some typical results will be given in a later paper. When $\kappa = 0$, Eq. (50) reduces to Eq. (48) with X_1 assumed zero. When $X = 0$, the limiting form of Eq. (50) is given by Eq. (49) for general values of κ and Y_1.

Let us next consider the value of the modified downwash amplitude W_b along the mid-chord axis of the blade. From Eqs. (43) and (46), it follows that:

$$W_b(0, Y_1) = \overline{W}_b(0, Y_1) - \frac{1}{4\pi}\int_{-1}^\infty \frac{\partial K}{\partial X} E(X, Y_1) dX \tag{51}$$

where $E(X, Y_1)$ is defined by Eq. (50). At points on the blade, $-1 < X < 1$, and it can be assumed that X is small compared to Y_1. Then, if we use the lifting line technique, it can be further assumed that the vorticity on the blade is concentrated along the mid-chord axis and that the wake is displaced forward to commence at mid-chord instead of the trailing edge. For compressible flow $\partial K/\partial X + i\nu K = 0$ in the wake and hence we may assume that for the displaced wake, $K(X) = K(1, Y)e^{-i\nu X}$, where $K(1, Y)$ represents the distribution of circulation over the blade. Equation (51) can then be replaced by the approximate form

$$W_b(0, Y_1) = \overline{W}_b(0, Y_1) - \frac{K(1, Y_1)}{4\pi}\left[E(0, Y_1) - i\nu \times \int_0^\infty e^{-i\nu X} E(X, Y_1) dX\right] \tag{52}$$

This corresponds to Eq. (39) which was obtained for incompressible flow conditions with $\nu = \omega$, $\kappa = 0$.

Equation (52) can be expressed in a different form when κ tends to zero. It follows from Eq. (16) that

$$4\pi W_b(0, Y_1) = \int_0^\infty \int_{-1}^\infty K(X, Y) \frac{\partial^2}{\partial Z_1^2}\left(\frac{1}{r}\right) dX dY - \int_0^\infty \int_{-1}^\infty K(X, Y) \frac{\partial^2}{\partial Z_1^2}\left(\frac{1 - e^{-i\kappa r}}{r}\right) dX dY \tag{53}$$

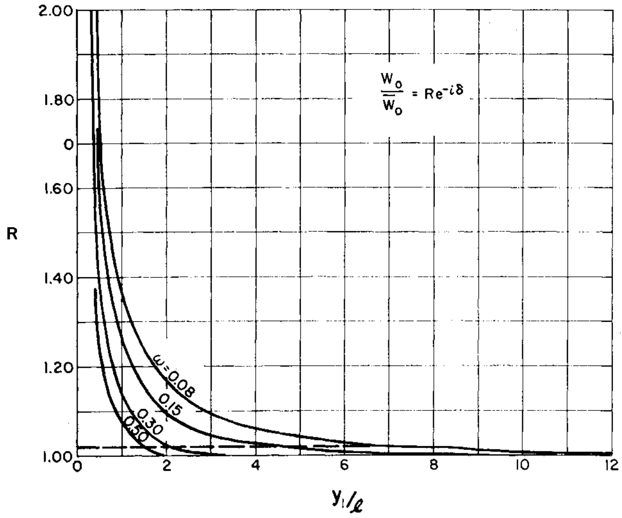

Fig. 5 Values of amplitude of $W_0/\overline{W}_0$.

and it can be shown[11] that the second term is of order $\kappa^2 \log_e \kappa$. It then follows that, to first-order accuracy in κ, this term can be neglected. The remaining expression is simply the formula for W_b as given by incompressible flow theory. However, since for the compressible case $\partial K/\partial X + i\nu K = 0$ in the wake, we find that according to the equivalent lifting line theory

$$W_b(0, Y_1) = \overline{W}_b(0, Y_1) + [K(1, Y_1)/4\pi Y_1] \times [1 - i\nu Y_1 F_c(\nu Y_1)] \tag{54}$$

This is the same as Eq. (39) for incompressible flow except that ω has been replaced by ν, the spanwise parameter Y_1 ($\equiv y/l$) by Y_1 ($\equiv \beta y_1/l$), and the W's are modified downwash amplitudes as defined by Eq. (14).

All the preceding analysis has been based on the 'half-model' representation of the flow with $K(1, Y)$ assumed to be constant over the blade with a value corresponding to that at the section $Y = Y_1$ at which the downwash is to be calculated. The theory can be extended, however, to include spanwise variation of $K(1, Y)$ as will be shown in the next section.

5. Extension of Theory

Let us assume that the general distribution $K(1, Y)$ over the blade illustrated in Fig. 2 is made up of layers of doublets of constant strength $\delta K(1, Y)$ extending from Y to ∞ where Y is variable. Then it follows from Eq. (52) that the modified downwash $\delta W_b(0, Y_1)$ due to a particular layer is given by

$$\delta W_b(0, Y_1) = \delta\overline{W}_b(0, Y_1) - \frac{\delta K(1, Y)}{4\pi}\left[E(0, Y_1 - Y) - i\nu \int_0^\infty e^{-i\nu X} E(X, Y_1 - Y) dX\right] \tag{55}$$

where $Y_1 > Y$. When $Y_1 < Y$, Eq. (55) is replaced by

$$\delta W_b(0, Y_1) = \frac{\delta K(1, Y)}{4\pi}\left[E(0, Y - Y_1) - i\nu \int_0^\infty e^{-i\nu X} E(X, Y - Y_1) dX\right] \tag{56}$$

By summing up the contributions from all the layers, the

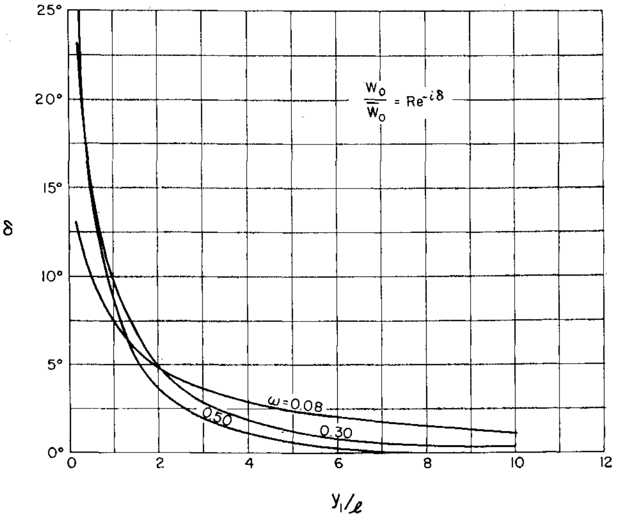

Fig. 6 Values of argument of $W_0/\overline{W}_0$.

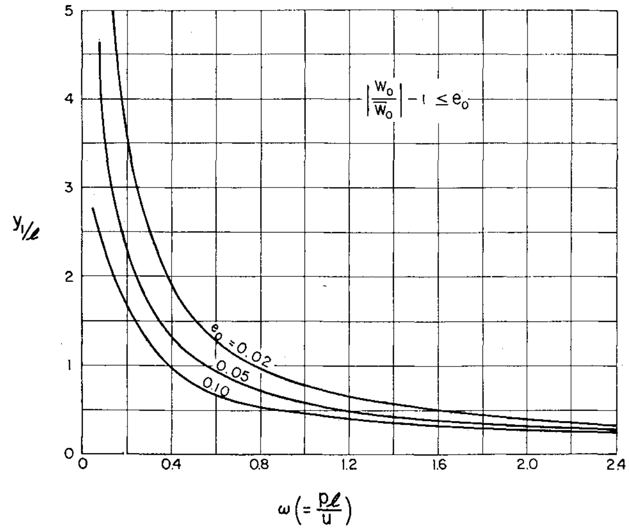

Fig. 7 Values of error, e_0.

following formula is obtained, namely

$$W_b(0,Y_1) = \overline{W}_b(0,Y_1) + \frac{1}{4\pi}\int_0^S \frac{aG(\nu,a)}{(Y - Y_1)}\frac{dK}{dY}dY \quad (57)$$

where $a = |Y - Y_1|$, $S = \beta s/l$, s being the blade span, and

$$G(\nu,a) = E(0,a) - i\nu\int_0^\infty e^{-i\nu X}E(X,a)dX \quad (58)$$

For low values of κ, the above expression can be replaced by

$$G(\nu,a) = -(1/a)[1 - i\nu aF_c(\nu,a)] \quad (59)$$

and Eq. (57) then reduces to the approximate formula

$$W_b(0,Y_1) = \overline{W}_b(0,Y_1) + \frac{1}{4\pi}\int_0^S \frac{1}{(Y_1 - Y)}[1 - i\nu aF_c(\nu,a)]\frac{\partial K}{\partial Y}dY \quad (60)$$

The above equation is an extension of Eq. (54) to allow for the variation in $K(Y)$ along the blade. It should be remembered that for the purposes of the present theory, the Mach number is assumed to be constant over the blade and that it has the value corresponding to its local value at $Y = Y_1$. The same applies to the parameter ν $(= \omega/\beta^2)$ and to all the other parameters that involve Mach number.

It is also possible to take account of the spanwise variation of K in the calculation of W_i as a result of the system of wakes under the blade. For a single blade, the corresponding value of $W_i(X_1,Y_1)$, based on the 'half-model' representation, is given by Eqs. (12) and (26) in the form

$$2\pi W_i(X_1,Y_1) = (\pi/D)K(1,Y)e^{i\nu(1-X_1)}[F_0(1) + F_1(1)] \quad (61)$$

where $F_0(1)$ and $F_1(1)$ are defined by Eq. (30). The actual downwash on the same basis is

$$w_i(x_1,y_1) = [k(l,y)/2d]e^{i\omega(1-X_1)}[F_0(1) + F_1(1)] \quad (62)$$

where F_0 and F_1 are independent of Mach number. The corresponding expression, when the spanwise variation of $k(l,y)$ is taken into account, is

$$w_i(x_1,y_1) = \bar{w}_i(x_1,y_1) - \frac{e^{i\omega(1-X_1)}}{2d}\int_0^S \frac{|y_1 - y|}{y_1 - y}F_1(1) \times \frac{\partial k(l,y)}{\partial y}dy \quad (63)$$

For a given $k(l,y)$ distribution and frequency parameter, $w_i(x_1,y_1)$ is independent of Mach number. This result was previously deduced in Ref. 1 when a two-dimensional model of the flow was used. The present analysis indicates that it is also true in three dimensions if curvature effects are neglected.

6. Discussion of Results

Some of the formulas developed in the previous sections have been used to illustrate the usefulness of the theory presented. From Eq. (42), values of the ratio $W_0/\overline{W}_0$ $(\equiv Re^{-i\delta})$ were calculated and they are shown plotted in Figs. 5–7. In Fig. 5, the amplitude R of this ratio is plotted against the distance y_1/l inboard from the tip for different values of the frequency parameter ω. The curve for $\omega = 0.5$ indicates that at a distance of one chord from the tip there is no significant difference between the value of W_0 as given by the "half-model" representation and $\overline{W}_0$, the two-dimensional value. For $\omega = 0.1$, the corresponding distance from the tip for which the difference would be negligible is about six chord lengths. The dotted line in Fig. 5 corresponds to $R = 1.02$ and the points where it intersects the ω curves give the values of y_1/l where the use of two-dimensional theory would lead to 2% error.

Similarly, Fig. 6, gives the values of the phase difference δ for different ω values. The curves show that $\delta < 2.5°$ for $\omega \geq 0.08$ when $y_1 > 4.5l$. From these results it may be concluded that the induced downwash due to the immediate wake of the blade can be calculated to sufficient accuracy by two-dimensional theory at all points of the blade located more than three chordlengths away from the tip, provided the frequency parameter at the tip is greater than 0.1.

In Fig. 7, an alternative form of plotting is used in which y_1/l is plotted against ω for various values of the difference e_0 $(\equiv |W_0/\overline{W}_0| - 1)$. The curve for $e_0 = 0.02$, for instance, corresponds to a 2% difference between results obtained by the use of the 'half-model' representation and the full two-dimensional model. For all points $(\omega,y_1/l)$ above the curve $e_0 = 0.02$, the error in using two-dimensional theory is therefore less than 2%.

The system of wakes under the rotating blade as represented in Fig. 2 also induces a downwash w_i at points on the blade. By the use of Eq. (30), the values of w_i/w_i were obtained for the case when $\epsilon(\equiv p/\Omega)$ was an integer. The results are shown plotted in Fig. 3 except that now $e_i \equiv |w_i/w_i| - 1$ and y_1/ld is plotted against ωd. The spacing between the wakes is dl and a direct comparison of Fig. 3 and Fig. 5 when $d = 1$, reveals that the correction is greater in the former case. This implies that in the calculation of w_i, tip vortex effects should be taken into account and the 'half-model' representation should be used. However, as far as the determination of airloads on the blade is concerned, a greater percentage error in w_i values might be permissible since $w_i < W_0$ in general. With regard to compressibility effects the analysis shows that w_i as a result of the infinite system of wakes beneath the blade is independent of Mach number of a given circulation $k(x,y,t)$ and frequency parameter value. Hence, it may be assumed that Fig. 3 is applicable for all Mach numbers. On the other hand, the values of W_b, the total downwash due to the vorticity on the blade and its immediate wake, are dependent on Mach number and may be determined by the use of Eq. (52). However, for low values of ν, a simpler approximate formula can be derived. This is given by Eq. (54) and is identical in form with Eq. (42), the equation for incompressible flow. Similarly, the curves given in Figs. 5 and 7 are also applicable provided ω is replaced by ν and y_1/l by $\beta y_1/l$. However, the condition that ν must be small may severely limit the usefulness of this approximation. The range of its validity is to be investigated in a further study of compressibility effects.

With both the two-dimensional and the "half-model" representation of the flow the downwash at any section y_1 is calculated on the basis of the assumption that the circulation at all sections is the same as that at the local section under consideration. In general, however, for higher modes of blade vibration than fundamental bending or twisting modes, it might be necessary to take into account more accurately the spanwise variation of the circulation. This could be done by using the "lifting line" type formulas that have been given in Section 5 of this paper. For compressible flow, Eq. (57) gives the values of the modified downwash amplitude W_b at any point $(0,Y_1)$ on the blade. An approximate form of the equation is given by Eq. (60). When $M = 0$, $\nu = \omega$ and $Y = y/l$, and the formula then reduces to a form that can be used for incompressible flow. Similarly, the spanwise variation of trailing vorticity in the system of wakes under the blade can be taken into account in the calculation of w_i. The appropriate formula is given by Eq. (63) which is applicable at all subsonic Mach numbers.

With the theory outlined, it should be possible to determine the loading on an oscillating blade on a three-dimensional basis for any prescribed motion. However, the analysis has been developed for rectilinear flow and the helical nature of the wake is not fully taken into account. It is possible, however, that at the higher frequencies of blade oscillation, curvature effects may not be important. Further work is planned to investigate this aspect and to compare airloads as given by different methods of calculation for a range of frequencies, Mach numbers, and modes of distortion for hovering flight.

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